From Classical Mechanics to Symplectic Geometry

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1. The situation in classical mechanics

Consider the motion of a particle with mass 1 in $\mathbb{R}^n(q)$ (called the *configuration space*), where q is the coordinate on $\mathbb{R}^n$ (the *position* of the particle), in the presence of a potential force

$$\Phi(q,t) = -\frac{\partial U}{\partial q}(q,t). \tag{1}$$

where $U(q,t)$ is the potential energy function. By Newton's second law (force is mass times acceleration), we have

$$\ddot{q} = \Phi(q,t) \tag{2}$$

We introduce now the new variable $p := \dot{q}$, which, since the partical has mass 1, we call the *momentum coordinate*, and consider now the *total energy function* of the particle

$$H(p,q,t) = \text{kinetic energy} + \text{potential energy} = \frac{1}{2}p^2 + U(q,t) \tag{3}$$

Then, using the usual trick for turning a second order ODE into a system of first order ODEs, (2) becomes:

$$\begin{cases} \dot{q} = p \\ \dot{p} = \Phi(q,t) \end{cases} \tag{4}$$

We now restate (4) in terms of F. First, taking $\dfrac{\partial}{\partial p}$ of both sides in (3) gives

$$\frac{\partial H}{\partial p} = p$$

and taking $\dfrac{\partial}{\partial q}$ of both sides gives

$$\frac{\partial H}{\partial q} = \frac{\partial U}{\partial q}(q,t)$$

and so

$$\frac{\partial H}{\partial q} = -\Phi(q,t)$$

thus, (4) reads as

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p}(p,q,t) \\ \dot{p} = -\dfrac{\partial H}{\partial q}(p,q,t) \end{cases} \tag{5}$$

which we call the *Hamiltonian equations*. This should be understood as a system of ODEs on the *phase space*, $\mathbb{R}^{2n}(p,q)$.

For our first bit of generalization, we forget the form of H in (3), and instead just ask that H behaves sufficiently nice to guarantee solutions to (5) for all $t \in \mathbb{R}$ (this translates to a condition on H at infinity). We continute to call H an energy function. Given such a function H, we can define the *flow*

$$h_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

which takes an initial condition $(p(0), q(0))$ to the corresponding solution $(p(t), q(t))$ at time t. Each h_t is a diffeomorphism of the phase space, and we call them *mechanical motions.*

1.1. Facts about mechanical motions.

Perhaps not the most classical way to think about it, but the Liouville theorem from classical mechanics basically states that the flow on phase space is incompressible. What this means is that given any closed surface S in phase space, after applying the flow to it for any time t the volume inside $h_t(S)$ is equal to that inside S. In terms of differential forms, the Liouville theorem reads:

Theorem (Liouville Theorem)**.** *Mechanical motions preserve the volume form*

$$Vol = dp_1 \wedge dq_1 \wedge \cdots \wedge dp_n \wedge dq_n,$$

i.e., $h_t^*(Vol) = Vol$.

This theorem was one of the main driving forces of Ergodic theory (volume preserving geometry). This theorem goes back over 100 years! Then, in the 1960's, along came Vladimir Arnold who proved something more subtle and general is actually true about mechanical motions:

Theorem (Arnold 1960's)**.** *Mechanical motions preserve the* 2*-form*

$$\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

Remark. *Note that, since* $Vol = \frac{1}{n!}\omega_0^n$ $(\omega_0^n = \underbrace{\omega_0 \wedge \cdots \wedge \omega_0}_{n})$, *this theorem implies the Liouville theorem.*

This result of Arnold led people to want to study the difference between mechanical motions and general volume preserving maps, and hence the field of symplectic topology was born! An example illustrating that there is indeed a difference is given by various *non-squeezing* results, e.g., one famous result by Gromov: given a cylinder $B^2(r) \times \mathbb{R}^{2n-2}$, there is no mechanical motion moving the ball $B^{2n}(R)$ into the cylinder if $R > r$. There is, however, a volume preserving map moving the ball into the cylinder (this should be easy to imagine). (Gromov's result is actually a little more general as it only asks for a *symplectic embedding.*)

2. To symplectic manifolds

To generalize to symplectic manifolds, we need to replace the phase space by a manifold that keeps key properties of the phase space. For simplicity, we replace it by a connected manifold.

Definition (Symplectic manifold)**.** *Let* M *be a* $2n$*-dimensional manifold endowed with a* 2*-form* ω *which is:*

- *Closed:* $d\omega = 0$
- *Non-degenerate:* $\omega^n \neq 0$ *at any point on the manfold*

The pair (M, ω) *is called a symplectic manifold.*

Remark. *An alternate way to state the non-degeneracy condition is: for any* $X \in \Gamma(TM)$, $X \neq 0$, *there is a* $Y \in \Gamma(TM)$ *such that* $\omega(X, Y) \neq 0$. *Yet another alternate way is that it gives an isomorphism of* TM *with* T^*M *via* $T_pM \ni X \mapsto \omega_p(X, \cdot) \in T_p^*M$.

2.1. **Examples of symplectic manifolds.**

Example (Trivial Example). *This is precisely the same as in the introduction.* $(\mathbb{R}^{2n}, \omega_0)$ *where* $\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$.

Personally, I like looking at this in complex coordinates better. Letting $z_j = p_j + iq_j$, *we have* $\mathbb{R}^{2n} \approx \mathbb{C}^n$ *and*

$$\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

Now, for the prototypical example of a symplectic manifold

Example (Cotangent Bundle). *Consider an* n*-manifold* V. *Its cotangent bundle,* T^*V, *is a* $2n$*-manifold which we call* M. M *comes equipped with a canonical* 2*-form,* $\omega = -d\lambda$, *which is decribed in local coordinates as follows:*
Given local coordinates $(x_1, ..., x_n)$ *on* V, *we get an associated set of local coordinates* $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ *on* T^*V. *In these coordinates,*

$$\lambda = \sum_{j=1}^{n} \xi_j dx_j, \qquad \omega = -d\lambda = \sum_{j=1}^{n} dx_j \wedge d\xi_j.$$

The pair (M, ω) *is a symplectic manifold.*

We can also define λ *invariantly (indepenent of coordinates) as follows:*
Given a point $p = (x, \xi) \in M$, *we define* λ *at* p *by*

$$\lambda_p = \xi \circ (D\pi)_p$$

where $\pi : T^*V \to V$ *is the usual projection map.*

Maybe a bit out of place with the flow of things so far, but here's an important theorem in symplectic geometry:

Theorem (Darboux's Theorem). *Let* M *be a* $2n$*-manifold with a symplectic form* ω. *Then* ω *is locally diffeomorphic to the standard form* ω_0 *on* $\mathbb{R}^{2n}$. *Alternatively said: near every point in* M, *one can choose local coordinates* (p, q) *such that* $\omega = \sum_{j=1}^{n} dp_j \wedge dq_j$.

What this theorem basically implies is that there is no such thing as "local symplectic geometry".

3. Everything rephrased in the symplectic language

In classical mechanics, energy determines the evolution of the system. For our energy function, we take a family of smooth functions, H_t, on M for t in some interval I. We can actually consider this family as a smooth function $H : M \times I \to \mathbb{R}$ by $H(x,t) = H_t(x)$. We call H a *time dependent Hamiltonian function.* Given an H_t, we get the *Hamiltonian vector field of* H_t, which we denote X_{H_t}, as the solution of the equation

$$\imath_{X_{H_t}} \omega = dH_t. \tag{6}$$

Letting t vary, we have a time dependent vector field. Now, the evolution of our system is determined by the *Hamiltonian equation*

$$\dot{x} = X_{H_t}(x).$$

In local coordinates (p, q) on M, the Hamiltonian equation takes on the following recognizable form:

$$\begin{cases} \dot{q} &= \frac{\partial H}{\partial p}(p, q, t) \\ \dot{p} &= -\frac{\partial H}{\partial q}(p, q, t) \end{cases}$$

Each Hamiltonian vector field X_{H_t} has a flow h_t (which we call a *Hamiltonian flow*). The individual h_t are called *Hamiltonian diffeomorphisms*, which replace our mechanical motions from earlier.

Remark. *Usually, we like to work with the time* 1 *map of flows, but since we can reparametrize flows, any Hamiltonian diffeomorphism can be realized as the time* 1 *map of some Hamiltonian flow.*

To satisfy the algebraists in the room, here are some nice properties about Hamiltonian diffeomorphisms:

Proposition. *The Hamiltonian diffeomorphisms form a group,* $\mathrm{Ham}(M,\omega)$*, with the following properties:*
(1) $\mathrm{Ham}(M,\omega)$ *is an infinite dimensional Lie group.*
(2) $\mathrm{Ham}(M,\omega)$ *is non-abelian.*
(3) *If M is a closed manifold, then* $\mathrm{Ham}(M,\omega)$ *is simple.*

Now, why did we choose the symplectic form to be closed? The main reason, I suppose, is simple because "it works", but here is a more satisfying reason: Hamiltonian diffeomorhpisms preserve the symplectic form. It suffices to show that $\mathcal{L}_{X_{H_t}}\omega = 0$ for all Hamiltonian vector fields to prove this claim. We have

$$\mathcal{L}_{X_{H_t}}\omega = \imath_{X_{H_t}} d\omega + d(\imath_{X_{H_t}}\omega) = \imath_{X_{H_t}} 0 + d(dH_t) = 0.$$

This is the general version of the theorem following the Liouville theorem. The nondegeneracy comes in by making sure that we can actually solve uniquely for X_{H_t}.

4. The Arnold Conjecture

In a sense, the first theorem of symplectic geometry is something from well before its genesis. This is a result originally stated by Poincaré in 1912 and proved by Birkhoff in 1913. It is sometimes referred to as *Poincaré's last geometric theorem*, but we will call it the *Poincaré-Birkhoff theorem*:

Theorem (Birkhoff 1913)**.** *Let A be an annulus, and let $\phi : A \to A$ be an area preserving homeomorphism which preserves the boundary circles, but twists them in opposite direction. Then ϕ must have at least* 2 *fixed points.*

Partially based on the Poincaré-Birkhoff theorem, in the 1960's Arnold conjectured the following (though not exactly in this language):

Conjecture. *A Hamiltonian diffeomorphism of a compact symplectic manifold should have at least as many fixed points as a function on the manifold must have critical points.*

Let's illustrate the power of this conjecture though an example. First, consider the Lefschetz fixed point theorem:

Theorem. *Let $f : X \to X$ be a continuous map from a compact triangulable space X to itself. Then, if $\Lambda_f \neq 0$, where*

$$\Lambda_f := \sum_{k\geq 0}(-1)^k \mathrm{Tr}\left(f_*|_{H_k(X,\mathbb{Q})}\right),$$

f has at least one fixed point. If we assume in addition that X is a smooth manifold, f is a smooth map, and all of the fixed points of f are nondegenerate (i.e. $\mathrm{graph}(f) \pitchfork \Delta_M$*), then the number of fixed points of f is at least Λ_f, and in addition*

$$\Lambda_f = \mathrm{graph}(f)\cdot\Delta.$$

(It's worth pointing out that Λ_f is a homotopy invariant.)

Let's consider a Hamiltonian diffeomorhpism h of the $2n$-torus $\mathbb{T}^{2n}$. Since Hamiltonian diffeomorphisms are homotopic to the identity, we have

$$\Lambda_f = \Lambda_{\mathrm{id}} = \sum_{k\geq 0}(-1)^k \mathrm{Tr}\left(id_*|_{H_k(X,\mathbb{Q})}\right) = \sum_{k\geq 0}(-1)^k b_k(\mathbb{T}^{2n}) = \chi(\mathbb{T}^{2n})$$

and so

$$\Lambda_h = \chi(\mathbb{T}^{2n}) = 0.$$

Thus, the Lefschetz fixed point theorem is useless here. However, all is not lost since, in 1983, Conley and Zehnder proved the Arnold conjecture for the $2n$-torus with the standard symplectic structure.

Theorem (Conley-Zehnder, 1983)**.** *A Hamiltonian diffeomorhpism of the standard torus* $\mathbb{T}^{2n}$ *has at least* $2n+1$ *fixed points. If all of the fixed points are nondegenerate, then it has at least* 2^{2n} *fixed points.*

As far as I know, in its full generality, the Arnold conjecture has yet to be proven. One case in which it is proven is when both classes $[\omega], c_1(TM) \in H^2_{dR}(M)$ vanish on $\pi_2(M)$. This was proven by Rudyak and Oprea. Since I should probably actually prove something, let me prove the Arnold conjecture in an easy case.

Lemma. *Suppose the Hamiltonian diffeomorhpism* $h : (M, \omega) \to (M, \omega)$ *is generated by a Hamiltonian function* H_t *which is independent of* t*. Then the Arnold conjecture holds.*

Proof. Fixed points of the Hamiltonian diffeomorphism h correspond to zeroes of the Hamiltonian vector field X_H. Zeroes of X_H in turn correspond to critical points of H. $\square$

References

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[3] Leonid Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms.* First edition, Birkhäuser, 2001.